INSTITUTE FOR MATHEMATICAL RESEARCH

Universiti Putra Malaysia Mathematical Olympiad 2016
UPMO 2016

Name : SOLUTIONS

Matric No. :

Faculty :

Date : 24 April 2016

Time : 9:00 am - 12:00 noon **Duration** : 3 hours

Instruction to Candidate

1. **Answer all questions.**

1. Let $\{x_n,\ n = 0, 1, 2, \dots\}$ be a sequence which is defined as $x_0 = k$ and $x_n = x_{n-1} + \frac{1}{n!},\ n = 1, 2, \dots,$ where k is a real number. Find $\lim_{n\to\infty} x_n$.

Solution:

$$\begin{aligned} x_n &= x_{n-1} + \frac{1}{n!} \\ &= x_{n-2} + \frac{1}{(n-1)!} + \frac{1}{n} \\ &= x_0 + 1 + \frac{1}{2!} + \cdots + + \frac{1}{(n-1)!} + \frac{1}{n}. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{n\to\infty} x_n &= a + \lim_{n\to\infty} \left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \\ &= a + e \end{aligned}$$

2. Let $\{x_1, x_2, ..., x_m\}$ be a set of non-zero vectors in $\mathbb{R}^n$ $(m \leq n)$ and A be a real n x n matrix such that $Ax_1 = x_1$ and $Ax_i = x_i + x_{i-1},\ i = 2, 3, 4, ..., m.$

   Prove that the set of vectors $\{x_1, x_2, ..., x_m\}$ is linearly independent.

Solution:

If, $m = 1$, clearly, statement is true. Let vector $x_1, \ldots, x_k (k < m)$ be linearly independent. Assume that for the vectors $x_1, \ldots, x_k$ $(k < m)$ be linearly independent. Assume that for the vectors $x_1, \ldots, x_k, x_{k+1}$,

$$c_1x_1 + \cdots + c_kxk + c_{k+1}x_{k+1} = 0 \tag{1}$$

for some numbers $c_1, \ldots c_{k+1}$ not all of which are zero.
Then,

$$A(c_1x_1 + \cdots + c_kx_k + c_{k+1}x_{k+1}) = 0$$
$$c_1Ax_1 + \cdots + c_{k+1}Ax_{k+1} = 0$$

$$c_1x_1 + c_2(x_2 + x_1) + \ldots + c_{k+1}(x_{k+1} + x_k) = 0 \tag{2}$$

Subtract(1) from (2) to obtain

$$c_2x_1 + \ldots + c_{k+1}(x_{k+1} + x_k) = 0.$$

Since the set $x_1, \ldots, x_k$ is linearly independent

3. Find the smallest number when divided by 3,4,5 and 6 will leave remainders 1,2,3 and 4 respectively.

Solution:

$$
\begin{aligned}
N &= 1 mod(3) = 2 mod(4) = 3 mod(5) = 4 mod(6) \\
N+2 &= 0 mod(3) = 0 mod(4) = 0 mod(5) = 0 mod(6) \\
N+2 &= 60 \\
N &= 60-2 \\
N &= 58
\end{aligned}
$$

$\therefore N = 58$

4. Let polynomial $P(x)$ satisfies the identity

$$x^{2016} - x^{2014} + x^{2013} - 1 \equiv (x^3 - x^2 + x - 1) \cdot P(x)$$

Find $P(1)$.

Solution:

$$(x-1)(X^{2015} + x^{2014} + x^{2012} + \cdots + 1) \equiv (x-1)(x^2+1)P(x)$$
$$\Updownarrow$$
$$x^{2015} + x^{2014} + x^{2012} + \cdots + 1 \equiv (x^2+1)P(x)$$

Substitute $x = 1$ then $2015 \equiv 2 \cdot P(1) \Rightarrow P(1) = \dfrac{2015}{2}$

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function that satisfies the condition

$$f(x+y) \equiv f(x) + f(y) + 2xy.$$

Find $f(2016)$ if $f(1) = -2016$.

Solution:

Differentiating the identity twice we get $f''(x+y) = f''(x)$, Therefore, $f''(x)$ is constant and $f''(x) = c_1$.
Let,

$$f'(x) = c_1 x + c_2$$
$$f(x) = c_1 \frac{x^2}{2} + c_2 x + c_3$$

$\therefore$

$$f(x+y) = c_1 \frac{(x+y)^2}{2} + c_2(x+y) + c_3$$
$$= c_1 \frac{x^2}{2} + c_1 xy + c_1 \frac{y^2}{2} + c_2 x + c_2 y + c_3$$

from $f(x+y) \equiv f(x) + f(y) + 2xy$
we obtain $c_3 = 0$ and $c_1 = 2$

Therefore,

$$f(x) = x^2 + c_2 x$$

However, $f(1) = -2016$
Hence,

$$f(1) = 1 + c_2(1)$$
$$-2016 = 1 + c_2(1)$$
$$c_2 = -2017$$

so $f(x) = x^2 - 2017x$
Finally,

$$\begin{aligned} f(2016) &= 2016^2 - 2017(2016) \\ &= 2016(2016 - 2017) \\ &= -2016 \end{aligned}$$

6. Evaluate the integral

$$\int_0^5 g(x)dx,$$

where $g(x)$ is the inverse function to $f(x) = x^3 + x$.

Solution:

Let

$$y = x^3 + x$$
$$dy = (3x^2 + 1)dx$$

$y = 0, x = 0$ and $y = 5, x^3 + x = 5$

Then

$$\int_0^a g(y)dy = \int_0^a x(3x^2 + 1)dx$$
$$= \int_0^a 3x^3 dx + \int_0^a x dx$$
$$= \frac{3}{4}a^4 + \frac{a^2}{2}$$

where $a^3 + a = 5$